TOPOLOGY-III
M.MATH-II
MIDTERM EXAM

Total 30 Marks

(1) Let $C = (C_k)$ be a free chain complex and let G be an abelian group. Let $Z_k \subseteq C_k$ and $B_k \subseteq C_k$ be the subgroups of cycles and boundaries, respectively. Let $i_k : B_k \to Z_k$ be the inclusion of boundaries into the cycles. Then show that

$$\text{Coker } i_k^* \cong \text{Ext}^1(H_k(C), G),$$

and

$$\text{Ker } i_k^* \cong \text{Hom}(H_k(C), G),$$

where $i_k^* : \text{Hom}(Z_k, G) \to \text{Hom}(B_k, G))$ is the induced map. (3 marks)

(2) Let G be an abelian group and let S^k be the standard k-sphere.
  (i) Compute the cohomology groups of S^k with coefficient in G.
  (ii) If $f : S^k \to S^k$ has degree d, then determine the map $f^* : H^k(S^k, G) \to H^k(S^k, G)$. (4 marks)

(3) Show that $\text{Ext}^1(\mathbb{Z}_n, \mathbb{Z}) \cong \mathbb{Z}_n$ and $\text{Tor}(\mathbb{Z}_n, \mathbb{Z}_n) = \mathbb{Z}_n$. (3 marks)

(4) Let $\mathbb{RP}^n$ be the real projective space.
  (i) Show that $H_k(\mathbb{RP}^n, \mathbb{Z}_2) \cong \mathbb{Z}_2$ for all $0 \leq k \leq n$.
  (ii) Compute the cohomology groups of $\mathbb{RP}^n$ in $\mathbb{Z}$ and $\mathbb{Z}_2$ coefficients. (4 marks)

(5) Consider the quotient map $f : \mathbb{RP}^2 \to S^2$ obtained by collapsing the subspace $\mathbb{RP}^1$ to a point. Determine the map

$$f_* : H_2(\mathbb{RP}^2, \mathbb{Z}_2) \to H_2(S^2, \mathbb{Z}_2)$$

and

$$f^* : H^2(\mathbb{RP}^2, \mathbb{Z}_2) \to H^2(S^2, \mathbb{Z}_2).$$

From this deduce that the splitting in the universal coefficient theorem for homology (and for cohomology) is not natural. (4 marks)

(6) Let X be obtained from $S^1 \vee S^1$ by attaching two 2-cells by the words a^2b^2 and ab. Compute the homology group of X with coefficient in $\mathbb{Z}$. (4 marks)

(7) Let X be the quotient space of S^2 under the identifications $x \sim -x$ for x in the equator S^1. Compute the homology groups $H_i(X, \mathbb{Z})$. (4 marks)

(8) Let X be a space such that $H_n(X, \mathbb{Z})$ is finitely generated for all n. Show that if $\widetilde{H}^n(X, \mathbb{Q})$ and $\widetilde{H}^n(X, \mathbb{Z}_p)$ are zero for all n and all p, then $\widetilde{H}_n(X, \mathbb{Z}) = 0$ for all n and hence, $\widetilde{H}^n(X, G) = 0$ for all n and G. (4 marks)